THE COMPATIBILITY INDEX G
CREATING AN INDEX OF CLOSENESS WITHIN WEIGHTED ENVIRONMENT

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ABSTRACT

This article addresses the problem of measuring closeness in weighted environments (decision-making environments). The article belongs to the field of mathematical modelling based in order topology. The relevance of this article is related with having a dependable cardinal measure of distance in weighted environments (order topology). Weighted environments is a no isotropic structure where the different directions (axes) may have different importance (weight) hence, there exist privilege directions. In this kind of structure is very important to have a cardinal reliable index, able to say how close or compatible is the set of measures of one individual with respect to the group (or to anyone other). Or how close is one pattern of behavior to another or in some special cases to assess how good a rule of measurement or index, built with any cardinal MCDM method is. Common examples of application of this is the interaction between actors in a decision making process (system values interaction), matching profiles, pattern recognition, and any situation where a process of measurement with qualitative variables is involved.

KEYWORDS:

Weighted environments, Measurement, Compatibility index G, Order topology.

1. INTRODUCTION

When using the concept of closeness come in mind immediately what means to be close, or in other words; when close really means close. Thus, when measuring closeness or proximity we should have a point of comparison (a threshold) that make possible to compare or decide if our positions, system values or priorities are really close. This is relevant issue since many people believe that the order of preference represent compatibility. For instance, is not necessarily true that two people (or group of people) with the same order of preferences have the same system value or decision-making priorities. Compatibility index G becomes a possible path to obtain consensus, without producing a big distance with the original system value (keeping compatibility alive). For our purposes, compatibility is defined as the proximity or closeness between vectors within a weighted space. [1] In this paper, it is shown a proposition for a compatibility index able to measure closeness in a weighted environment. Thus, able to assess pattern recognition, like medical diagnosis support measuring the degree of closeness between disease-diagnosis profiles, Buyer-Seller matching profiles; measuring the degree of closeness between house buyer and seller projects, or employment degree of matching; measuring the degree of closeness between a person's profile with the desired position profile; in curricula network design. Conflict Resolution; measuring closeness of two different value systems (the ways of thinking) by identify and measuring the discrepancies, and in general measuring the degree of compatibility between any priority vectors in cardinal measure bases (order topology).[1,2]. This index is also relevant, since it makes possible to do measurement of proximity in weighted environment using relative absolute ratio scale as priority vectors. This is important because many MCDM methods work in this kind of environment. Also, the investigation in this field is very rarely, and most of the studies are conducted using the square distance, Euclidean Norm, Log of Max over Min, dot product or Hadamard product. But, those indices do not take into account the crucial fact that we are working in a weighted environment, using a relative absolute ratio scale and not an interval scale (scale of differences), where the zero value may not be defined.

1.1 Content and order of the paper:

The paper is ordered presenting first: the literature review (2); with some theory of distance (measurement) and closeness for different points of view, from statistical and from set theory for measuring distance and similarity. Then, in (3) is presented the Garuti´s compatibility index G, the incompatibility index is introduced, and some analogies between G and distance. Then, in (4) a necessary threshold is presented, the threshold allows to establish "when close really means close" in weighted environments. This point responds to the necessity of any measurement index to have a threshold in order to make a correct interpretation of the value. In (5) two relatively simple examples are developed, each one presenting a different application for index G. The first example is about using index G to questioning if the order of choice should be "a must" to say if two rankings are compatible (close) or not. And the second one is about using the G index as a quality test, in this case testing if Saaty's consistency index is a good index or not. Finally, in (6) are shown the conclusions, where one global conclusion and several specific conclusions are presented. First; the necessity of a compatibility index to correctly evaluate the distance and/or similarity within weighted environments. And second; the different fields where the compatibility index G has been successfully applied.

2. LITERATURE REVIEW

In metric topology,[2] the particular function of distance D(a,b) is used to assess the closeness of two points a, b as a real positive function that keeps 3 basic properties:

1.- $D(a,b) > 0$ and $D(a,b) = 0$ iif $a=b$ (definition of zero distance)

2.- $D(a,b) = D(b,a)$ (symmetry)

3.- $D(a,b) + D(b,c) \geq D(a,c)$ (triangular inequality)

The general function of distance used to calculate the separation between two points is: $D(a,b)= \lim_{n\to k} (\Sigma_i(a_i-b_i)^n)^{1/n}$ (i=1,...,n; n= space dimension).

When applying different values of k, different Norms of distance appear:

For k = 1, then: D(a,b)= Σ_iAbs(a_i-b_i). Norm1, absolute Norm or path Norm; this Norm measure the distance from *a* to *b* within a 1D line, "walking" over the path, in one line-dimension.

For k = 2, then: D(a,b)= $[(\Sigma_i(a_i-b_i)^2)]^{1/2}$. Norm2 or Euclidean Norm, this Norm measure the distance from *a* to *b,* within a 2D plane (X-Y plane) getting the shortest path (the straight line).

For k **= +∞**, then: D(a,b)= Max_i (abs (a_i-b_i)). Norm ∞ or Norm Max; this Norm measure the distance from *a* to *b* within a ∞D hyperplane, getting the shortest path (the maximum coordinate) from all the possible paths.

In the field of statistics we may note an interesting case of distance calculation which is known as distance of Mahalanobis (1936), which meets the metric properties showed before. This distance takes into consideration parameter of statistics like deviation and covariance, (which can be assimilated to concepts of weight and dependence in the AHP/ANP world). Its formal presentation is:

$d_m(x,y) = \sqrt{(\mathrm{X}-\mathrm{Y})\sum^{-1}(\mathrm{X}-\mathrm{Y})}$ With Σ^{-1} the matrix of covariance between X, Y.

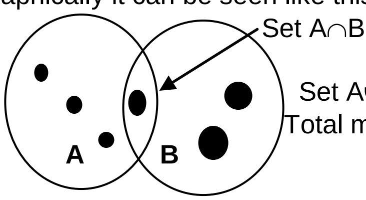
But, for a more simple case (without dependence), this formula can be written as:

$d_2(x_1,x_2) = \sqrt{(\frac{(x11-x12)}{\sigma 1})^2 + (\frac{(x21-x22)}{\sigma 2})^2}$ or $d_e(X_1, X_2) = \sqrt{(X_1-X_2)^T S^{-1} (X_1-X_2)}$,

with S^{-1} the diagonal matrix with the standard deviation of variables X, Y.

Is interesting to see that the importance of the variable (to calculate distance), is depending of the deviation value (bigger the deviation smaller the importance), this is saying that the importance of the variable is not depending of the variable itself, but in the level of certainty on the variable. By the way, is this statement always true?. In the field of Botanic, we may find another beautiful formula to measure the concept of similarity among species, this time coming from the Set Theory domain. Is the so called "Jaccard index", from Paul Jaccard, Jaccard (1901). Which, circa 1912 defined in a very simple way, that the similarity of two sets of objects is given by its ratio of intersection and union, that is: J = (A∩B) / (A∪B), which can be write as: (ΣMin(A,B)) /(ΣMax(A,B)). Considering that the minimum quantity of elements present simultaneously in two sets is given by its intersection and the maximum by its union.

Graphically it can be seen like this:



Set A∪B = Set A + Set B – Set A∩B
Total matching when: A∪B = A∩B (Min=Max)

An approximate vector expression of Jaccard index (using the dot product expression), can be written as: J = (A•B) **/** (A•A + B•B – A•B), considering that dot product represent the intersection of two set (vectors) A and B. If A and B are parallel vectors, then there is a total intersection, when they are perpendicular vectors then there is null intersection. (The subtraction in the denominator is to avoid the double counting of elements). Thus, the intersection is a way to measure the degree of projection that two vectors may have.

Here is an important coincidence with the compatibility index G, since both approach try to measure the degree of matching of different groups and both have a strong relation with the Min and Max functions and its ratio. The big difference between them, begin when one try to consider the concept of importance of the elements of the set (the weighted space concept), where the main question is: what happen when the elements in the set (A and/or B) have different importance or weight? Which, by the way, represent the general case in decision making domain. We take these approach into discussion, since factors as weight and dependence are in the bases of AHP and ANP structure [4,5]. But, instead to have to understand and deal with probabilities and statistics (which by the way are not easy to build and later interpret), the idea here is to apply the natural way of thinking of human being which is based more on priorities than in probabilities. Indeed, we can manage the same information in a more comprehensive, complete and easy to explain form combining AHP/ANP with compatibility index *G*, and working with priorities, avoiding the needs of collecting big databases or have to understand and interpret complex statistic functions. (By the way, priorities can include probabilities but not vice versa).

Thereby, the MCDM approach through AHP/ANP method gives a very nice tool for our investigation and treatment of the knowledge and experience that experts possess in their different fields, and at the same time staying within the decision making domain (order topology domain), avoiding of building huge and costly databases in where the knowledge about the individual behavior is lost.

3. THE GARUTI'S COMPATIBILITY INDEX (G)

In order topology measurement deals with dominance between preferences (intensity of preference), for instance: D(*a,b*)=3, means that dominance or intensity of preference of "*a*" over "*b*" is equal to 3, or that, *a* is 3 times more preferred than *b*. When talking about preferences a *relative absolute ratio* scale is applied. *Relative;* because priority is a number created as a proportion of a total (percent or relative to the total) and has no needs for an origin or predefined zero in the scale. *Absolute;* because it has no dimension since it is a relationship between two numbers of the same scale leaving the final number with no unit. *Ratio;* because it is built in a proportional type of scale (6kg/3kg=2). [2]. So, making a general analogy between the two topologies, one might say that: *"Metric Topology is to Distance as Order Topology is to Intensity".*[2,6]

An equivalent concept of distance is presented in order to make a parallel between the three properties of distance of metric topology [1,2]. This is applied in the order topology domain, considering a compatibility function *(Eq.1)* similar to distance function, but over vectors instead of real numbers.
Consideration:

A, B, C are priority vectors of positive coordinates, and $\Sigma_i a_i = \Sigma_i b_i = \Sigma_i c_i = 1$.
G(A,B) is the compatibility function expressed as:

$$\boldsymbol{G(A,B) = ½ \sum\left((ai + bi)\frac{Min(ai,bi)}{Max(ai,bi)}\right)} \quad \text{(Eq.1)}$$

This function presents: [1, 2, 5, 6]

1.- 0 ≤ *G*(A,B) ≤ 1 *(Non negative real number)*

The compatibility function *G*, returns a non-negative real number that lays in the 0 - 1 range. With *G*(A,B)=0, if A and B are perpendicular vectors (A⊥B), and represent the definition of total incompatibility between priority vectors A, B.(A□B=0). Also, *G*(A,B)=1, if A and B are parallel vectors, (A=B for normalized vectors), and represent the definition of total compatibility between priority vectors A and B. (A∘B = 1)

2.- *G*(A,B) = *G*(B,A)*(Symmetry)*

Symmetry condition, the compatibility measured from A to B is equal to the compatibility measured from B to A. Easy to proof, just interchanging A for B and B for A in the compatibility function *G*.

3.- *G*(A,B) + *G*(B,C) ≥ *G*(A,C) *(Triangular inequality)*

4.- If A***C***B and B***C***C =><= A***C***C (*Non transitivity of compatibility*): If A is compatible with B and B compatible with C, does not implies that A is necessarily compatible with C.

For property 3, is easy to prove that if A, B and C are compatible priority vectors (i.e. 0.9 ≤ G_i ≤ 1.0 for A, B, C), then property 3 is always satisfied. But, this property is also satisfied for the more relaxed (and interesting) condition where only two of the three vectors are compatible. For instance, if A is compatible with B (*G*(A,B)≥0.9) and A is compatible with C (*G*(A,C)≥0.9), or some other combination of A, B and C, then condition 3 is also satisfied. This more relaxed condition allows compatible and non-compatible vectors to be combined while property 3 is still satisfied.

This situation can be geometrically viewed in the FIGURE 1:

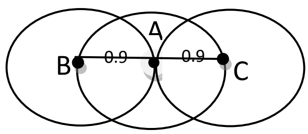


FIGURE1: Maximum circle of compatibility for position A, related to B and C [2]

FIGURE 1, is showing the compatibility neighborhood for A, in relation with B and C, with its minimum compatibility value of 0.9 represented by the radius of the circle. (In the center the compatibility reaches its maximum value of 1.0). Thus, *G*(A,B)=*G*(A,C)=0.9 is representing the minimum compatibility point, or the maximum distance for positions B and C to still be compatible with position A. Of course, *G*(B,C)<0.9 that represents a non-compatible position for points B and C. Notice that property 3,*G*(A,B) + *G*(B,C) ≥ *G*(A,C)is still valid, indeed any combination that one can be made will keep the inequality satisfied since if C gets closer to A (increasing the right side of the equation), then *G*(B,C) will also grow. The extreme case when C is over A, (*G*(A,C)=1.0) then

G(B,A)+*G*(B,C)=0.9+0.9=1.8>1.0 keeping the inequality satisfied. [2]. It is also possible to define the incompatibility function as the arithmetic complement of the compatibility:
Incompatibility = 1 – Compatibility.

Thus: Incompatibility is equivalent to (1 – *G*). *By the way, the incompatibility concept is more close to the idea of distance, since the greater the distance the greater the incompatibility.* [1, 2, 5, 6]

Two simple examples of this parallel between D(x,y) and *G*(X,Y) are given. But first, to make D and G functions comparable, absolute distance D must be transformed into relative terms as a percent value since the priority vectors are normalized vectors for the G function. Thus, the maximum possible value for D^1 (Norm1) is 2 and for D^2 (Norm2) is$\sqrt{2}$, while performing the ratios with respect to the maximum possible value and obtaining D in relative terms as percent of the maximum value.

For the first example, two different and very different vectors A and B with coordinate: {0.3, 0.7} with {0.7, 0.3} and {0.1, 0.9} with {0.9, 0.1} are considered. Considering also Incompatibility= 1–Compatibility or 1 –*G*(A,B). Then, compatibility between A and B is shown by *G*(A,B) *(Real positive value laying in 0-1 range).*Incompatibility between A and B is shown by 1– *G*(A,B) *(Real positive value laying in 0-1 range).*

TABLE 1A shows the results of applying D^1, D^2 and (1-*G*) functions.

A,B Coordinates	D^1(a.b) Distance A-B in Norm1 (normalized)	D^2(a,b) Distance A-B in Norm2 (normalized)	Incompatibility= 1-*G*(A,B)
A={0.3, 0.7} B={0.7, 0.3}	0.8/2= 0.4 (40%)	0.4√2/√2= 40%	1-(0.3/0.7)= 57%
A={0.1, 0.9} B={0.9, 0.1}	1.6/2= 0.8 (80%)	0.8√2/√2= 80%	1-(0.1/0.9)= 89%

TABLE 1a: Evaluating distance and incompatibility for non-similar set of coordinates

FIGURE 2, shows in 2D Cartesian axes how far (incompatible) is A from B in both cases. Notice that for the case represented in brown (for D=80% and G=89%), vectors A and B are (geometrically) almost in a perpendicular position (A relative to B). [2]

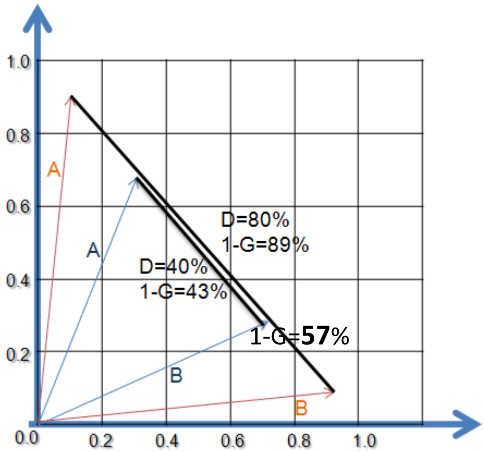


FIGURE 2: Two examples for distance and compatibility functions for far and very far A and B

Next, using the same procedure, we compare for similar and very similar A, B vectors with coordinates: {0.3, 0.7} compared with {0.4, 0.6} and {0.10, 0.90} compared with {0.11, 0.89}. Table 1b shows the results of applying D^1, D^2 and 1-*G* (Incompatibility = 1- Compatibility).

A,B Coordinates	D^1(%)	D^2(%)	Incompatibility= 1-*G*(%)
A={0.30, 0.70} B={0.40, 0.60}	0.2/2= 0.1 (10%)	0.1√2/√2= 10%	1-0.820= 18%
A={0.10, 0.90} B={0.11, 0.89}	0.02/2= 0.01 (1%)	0.01√2/√2= 1%	1-0.981= 1.9%

TABLE 1b: Calculating distance and incompatibility for similar set of coordinates

The trend of the results for D and *G* functions is the same in both cases, when increasing the distance or making vectors more perpendicular and when decreasing the distance or making the vectors more parallel. This is an interesting parallel to these concepts and their trends, considering that different concepts (distance and incompatibility in different ratio scales) are being used. [2]

Possible applications:

There are many different applications for index G, next a summary with all possibilities that G may have:

- **Compatibility of systems value:**

G is an index able to be used in social & management sciences to measure compatibility of group decision-making (DMs) intra and inter groups. The expression of G for this case is: G(DM1-DM2), which means level of compatibility (closeness) between DM1 and DM2. With DM1 and DM2 the decision's metric of each decision maker.

- **(Compatibility for quality test):**

G can help to assess the quality of a built decision metric. As presented in point 7.2, G may help to evaluate the quality of any new metric based in a ratio scale. The result is achieved comparing the new metric with some standard or with an already known result.

- **Profiles alignment:**

G can help to establish if two different profiles are aligned. In general, is not an easy task to know if two complex profiles are aligned, especially in case that the profiles are complex with many variables with different importance and behaviors. This is the case when try to measure the degree of matching between a medical diagnose and a list of diseases, or the degree of matching between a sale project and its possible buyers and many other similar cases.

- **Compatibility for Comparability:**

G can help to establish if two different measures are or not comparable, one relevant point when compare numbers from different outcomes is to know if those numbers are comparable or not. For instance, if I know that the impact of strategy A is 0.3 and the impact of strategy B is 0.6, I cannot say that strategy A has twice impact than strategy B; at less, both strategies were measured with exactly the same rule. But, for many reasons, sometimes that is not possible. In that case, we need to know if the rules of measurements are compatible among them. If so, it is possible to compare both numbers, if they are not compatible then we cannot.

- **Compatibility for sensitive analysis & threshold:**

G can help to establish the degree of membership or the trend for membership (tendency) of an alternative. The idea is equivalent to the classic sensitive analysis when making small changes in the variables. The change resulting in the G value (before and after the sensitive analysis), would show where the alternative is more likely to belong (trend of belonging).

4. GENERATING A THRESHOLD FOR COMPATIBILITY INDEX G:

To answer the initial question (*when close really means close*), first it is necessary to have a reliable index of compatibility. However, that is not sufficient it is also necessary a second condition a limit or threshold for the index. For useful purpose, it is necessary to have a limiting lower value (minimum threshold) to indicate when two priority vectors are compatible or close to be compatible, in order to define precisely when close really means close. We have four different ways to define a minimum threshold for compatibility: [2]

First: compatibility is ranged between 0 to 100 percent ($0 < \cos\alpha < 1$), being 100% the case of total compatibility (represented by parallel vectors). It is reasonable to define a value of 10% of tolerance (1/10th of 100%) as a maximum threshold of incompatibility to consider two vectors as compatible vectors (which means a minimum of 90% of compatibility to consider two vectors compatible). This explanation is based on the idea of one order of magnitude for an admissible perturbation for measurement

Second: In table below (table2), is presented a sequence of 2, 3, 4 and 5 dimension vectors, the first or initial vector is obtained as an isotropic flat space situation, it means equal values (1/*n*) in each coordinate (no privileged direction in the space); the second one is a vector obtained perturbing (adding or subtracting) 10% on each coordinate, creating "small crisps" or little privileged directions, then the incompatibility index is calculated with the 5 different formulas, (the reason to use all formulae, is because we are working on a near flat space (no singularities), where every formula works relatively well).

	CASE SENSIBILITATION TABLE For 2-3-4 and 5D Homogeneous Vectors (perturbing flat space to near flat space)									
Dim	Coordinates Perturbing 10% the initial vector of coordinates and normalizing					Hadamard (Saaty's Index) (%)	*G* *(%)*	Hilbert (%)	IVP (%)	Norm 1 (%)
2D	0.500	0.500	*Initial*			1.010	9.52	8.715	1.01	5.00
	0.450	0.550	*Pertur*							
3D	0.333	0.333	0.333	*Initial*		0.898	8.19	8.715	0.89	4.30
	0.354	0.290	0.354	*Pertur*						
4D	0.250	0.250	0.250	0.250	*Initial*	0.910	9.52	8.715	1.01	5.00
	0.275	0.225	0.275	0.225	*Pertur*					
5D	0.200	0.200	0.200	0.200	0.200	0.969	8.96	8.715	0.97	4.70
	0.215	0.176	0.215	0.176	0.215					

TABLE 2: Defining a possible threshold of 10% for *G* function

When looking the outputs for incompatibilities, it is possible to observe a good response for everyone (equal or less than 10%), with *G* and Norm1 circa 10% and 5% as upper bound in every case.

Third: In the figure below (FIGURE 3), a simple test was run over an Excel spreadsheet, using the common area example of AHP. The result (the importance of the area of the figures) can be calculated precisely with the typical geometric formulas and then normalizing its values to obtain the exact priorities as a function of the size of their areas. Doing this way, it is possible to have a reference point of the element values (the right coordinates for the actual area vector).

The next step is perturbing the actual area values by +/- 10% producing a new vector of areas, finally over these two vectors (actual and perturbed) is applied the *G* function to measure their compatibility obtaining a value of 91.92% (or 8.08% of incompatibility), this result is very close to the standard error deviation calculated as: Σ *Abs*(perturbated-actual)/actual=10%, this is showing that 90% might represent a good threshold, considering that the difference between both outputs is related with the significant fact that this numbers are not just numbers but weights. In fact, we are working with intensities that reflect sensations and experiences, not physical numbers.

AREA EXAMPLE

	Perturbation 10% up&down	Normalized	Actual Value	Error Deviation	
Circle [A]	0,5225	0,505	0,475	10,00%	0,0475
Square [C]	0,2088	0,202	0,232	10,00%	0,0151
Diamond [D]	0,1661	0,160	0,151	10,00%	0,0093
Rectangle [E]	0,0837	0,081	0,093	10,00%	0,0049
Triangle [B]	0,0539	0,052	0,049	10,00%	0,0232
	1,035	1,000	1		10,00%

INDEX OF COMPATIBILITY (*G*)= 91,92%

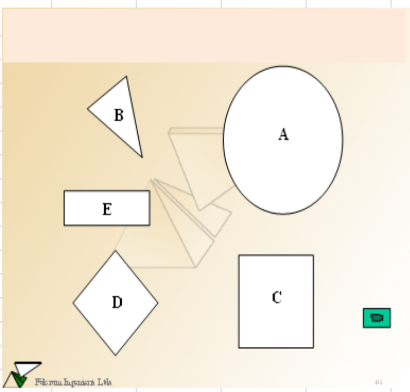
INDEX OF INCOMPATIBILITY= 8,08% Compatible

FULL RANDOM AREA

Random values	Normalized	
32,41%	15,29%	
46,31%	21,84%	
28,38%	13,39%	
82,37%	38,85%	
22,55%	10,64%	
2,1202	1,0000	
G (random)=	53,27%	
Incompatibility=	46,73%	>10%

SORTED RANDOM AREA

Random with Order	Normalized
0,700	59,83%
0,210	17,95%
0,150	12,82%
0,060	5,13%
0,050	4,27%
1,1700	1,0000
G (random)=	78,35%
Incompatibil	21,65%



	Compatibility Without Order	Compatibility Keeping Order	Nº Experiment
	33,57%	75,25%	1
	65,71%	58,08%	2
	64,54%	70,68%	3
	48,06%	85,37%	4
	52,57%	58,88%	5
	45,54%	82,08%	6
	50,62%	84,40%	7
	48,63%	91,98%	8
	28,44%	86,40%	9
	44,51%	74,99%	10
	59,94%	78,88%	11
	35,71%	87,95%	12
	70,23%	75,33%	13
	43,56%	75,79%	14
	61,07%	77,77%	15
Average	50,18%	77,59%	

Conclusions:

If the average of compatibilities generated by a random function is about 50%, and 78% when considering the order of the figures, then the limit of 90% seems to be a good threshold.

FIGURE 3: Possible Threshold of 10% for index *G*

Fourth: the last way to analyze the correctness of 90% for threshold was carry out, it consist in working with a random function and filling the area vector with random values and calculating *G* for every case. The goal is to generate an average *G* for the case of full random values for the areas ("*full* *random*” means without any previous order among the areas, like figure A is clearly bigger than figure B, and so on). And again producing random values but this time keeping the correct order among the figures (imitating the behavior of a rational DM), and once again generating an average *G* for this case, then both results are compared against actual values.

The average value of *G* for 15 tries in the first case (keeping no order), was around 50% of compatibility and 78% for the second case (keeping the order among the 5 figures). Both results shows that limit of 90% might be a good threshold, in the first case the ratio between threshold and the full random *G* is almost twice bigger 1.8 (0.90 over .50), keeping the 0.90 compatibility threshold far from random responses.

In the second case (threshold over sorted figures), the ratio is much more closer (as expected to be), with a value of 1.16 (0.90 over 0.78), saying that order may help to improve compatibility but is not enough, it needs to consider the weights (it means not just the preference but the intensity of the preference) which is related to the values of the elements that belong to the vector, as well as the angles of both vectors point to point (geometrically viewed as profiles).

Of course, this test should be carried for a larger number of experiments to have a more reliable response. A second test conducted for 225 experiments (15 people making 15 experiments each), has shown more or less the same initial results for average *G* value in both case with and without order (±0.78 & ±0.50).

Next, is presented a table with the meaning of ranges of compatibility in term of index G and its description:

Degree of Compatibility	Compatibility value range (G%)	Description
Very High	≥ 90%	Very high compatibility Compatibility at cardinal level (Compatible vectors)
High	85 – 89.9	High Compatibility (Almost compatible vectors)
Moderate	75 – 84.9	Moderate compatibility (compatibility only at ordinal level)
Low	65 – 74.9	Low level of compatibility
Very Low	60 – 64.9	very low compatibility (Almost incompatible vectors)
Null (random)	< 60%	Random level of compatibility (Incompatible vectors)

TABLE 3: Ranges of compatibilities and its meaning

Finally, another interesting way to illustrate the 90% as a good threshold for compatibility (the one I like the most), is the pattern recognition factor. Compatibility is the way of measure if a set of data (vector of priorities or profile of behavior) correspond to a recognized pattern or not. For instance, in the example of medical pattern recognition, the diagnose profile (the pattern) is built with the intensity values of sign and symptoms that correctly describe the disease. Then, it is compared with the sign and symptom gathered from the patient, when these two profiles have circa 90% or more of matching, then the physician was confident to say that the patient has the described disease.

When the patient and disease profiles shown a matching level between 85-90%, then the physicians in general agreed with the diagnoses offered by the software. However, when the *G* values was below 85% (between 79-84%), then the doctor sometime found trouble to recognize if the new sign & symptoms (the new patient's profile) was corresponding or not to the disease initially offered (he used to say that has no conclusive information). Finally, when the matching value (the *G* index) was below 75% the physician was not able anymore to clearly recognize in the patient's profile the disease initially offered.

Notice: these new profiles were built artificially, changing some values of sign and symptoms in an imaginary patient profile, in order to achieve matching values of 90%, 85%, 80% and so on, the intention was to evaluate when an experimented (and knowledgeable) doctor change his perception (mostly based on his pattern recognition ability).

Thus, two vectors may be considered compatible (similar or matching patterns) when ***G*** is greater or equal 90% with great certainty and confidence. In addition, values between 85-90% have in general a good chance to be correct (this is, to have a good level of certainty).

5. TWO SIMPLE APPLICATIONS OF COMPATIBILITY INDEX G.

5.1 First example. Is the order of choice a must?

We use to say that under the same decision problem, two compatible persons should make similar decisions. However, what do we mean when we say, "two compatible people should make similar decisions" [6,7]. It means that they should make the same choice?

Look to the following case:

Two candidates: A, B for an election. Three people, P1: choose candidate A, P2 & P3: choose candidate B. P1 & P2: are moderate people, thus their intensity of preference for the candidates are for person 1: 55-45, for candidate A, and person 2: 45-55 for candidate B.

By the other hand, P3 is an extreme person, thus his intensity of preference is 5-95 for candidate B.

Is really P3 more compatible with P2 than P1 just because P3 make the same choice of P2? (Both have the same order of choice voting by candidate B). It seems that the order of choice is not the complete or final answer.

In the other hand, we know that in order topology, metric of decision means intensity of choice (degree of dominance of A over B). So, compatibility is not related only with the simple order of choice, is something more complex and systemic, it is related with the intensity of choice.

Let`s see the next numerical example (Table 4).

Suppose three people having equal and different order of choice and its related priority vectors (intensity of choice):

Person 1 (P1)		Person 2 (P2)		Person 3 (P3)	
Order of Choice	Intensity of Choice	Order of Choice	Intensity of Choice	Order of Choice	Intensity of Choice
1º	0.364	3º	0.310	º1	0.501
2º	0.325	2º	0.325	2º	0.325
3º	0.311	1º	0.365	3º	0.174
		Order totally Inverted with P1		Order equally with P1	

TABLE 4: Comparing intensities and order of choices of 3 people

As we can see from TABLE 4, the order of P1 is the same than the order of P3 and different of P2

Order(P1) = Order(P3) ≠ Order(P2) (inverse order actually)

Considering just the above information, may we say that P1-P3 are closer than is P1-P2? Making the numbers (calculating G for both combination P1-P3 and P1-P2) we found:

G(P1;P2)= 0.9 (≥90%), which implies that P1 and P2 have compatible choices (very high compatibility).

G(P1;P3)= 0.77 (<90%), which implies that P1 and P2 have non compatible choices (moderate to low).

This is a very interesting result, considering that P2 have a complete inverted order of choice compared with P1. Yet, they are compatible people.

By the other side, P1 and P3, which have the same order of choice, are not compatible.

This example shows that is very important to measure the degree of compatibility (alignment) in a reliable way. Because is this measure which will tell us the real level of compatibility between people and not necessary their order of choice.

Curious note: in Alice in wonderland of Charles Lutwidge Dodgson (Lewis Carroll) is a phrase saying: "I tell you; sometimes 1-2-3 might look more like 3-2-1 than 1-2-3". [6,7]

5.2 Second example. Mixing consistency and compatibility indices in a metric quality test drive:

A different and interesting application of G is possible when is used to check the quality of a created metric or index.

When is possible to compare a metric obtained with some method with the expected or actual metric, then the compatibility index G represent a great tool to test and verify the quality of the created metric.

Suppose for instance, we want to measure the quality metric of the following example.

The Problem (the criticism):

Setting a problem (a criticism made by some critic person), about the quality of the consistency index in pair comparison matrices in the AHP (Saaty's Index of consistency), Saaty (2001, 2010)

The hypothetical critic says; *"The index of consistency of AHP (Saaty's index) is wrong, since it may let pass some values (comparisons) that are not acceptable by the common sense".*

The Example:
Suppose three equal bars of same long like the ones in FIGURE 4.1:

FIGURE 4.1: Bar length

Of course, the correct pair-comparison matrix (PC matrix) for this situation is the following (consistent) comparison matrix:

	A	B	C	
	1	1	1	A
A=	1	1	1	B
	1	1	1	C
w=	0,3333	0,3333	0,3333	

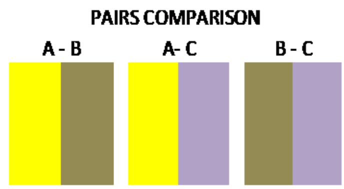


FIGURE 4.2: Bar comparisons 1

The obvious (correct) priority vector "w" is {1/3, 1/3, 1/3}, with 100% of consistency (CR=0). Since they are all equally long.

Suppose now that (due to some visualization mistake), the new appreciation about the bars is (FIGURE 4.3)

	A	B	C	
	1	1	2	A
A=	1	1	1	B
	1	1	1	C
w*=	0,4126	0,3275	0,2599	

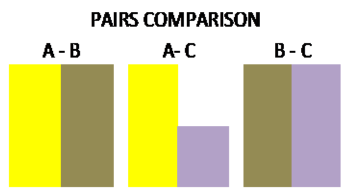


FIGURE 4.3: Bar comparisons 2

The new (perturbed) priority vector w* is {0.4126, 0.3275, 0.2599}, with CR = 0.05 (95% of consistency), which according to the theory is the maximum acceptable CR for a 3x3 comparison matrix.

The critic claim that the A-C bar comparison has a 100% of difference (100% of error), which is not an acceptable or tolerable error (easy to see even at naked eye).

Also, the global error (deviation) in the priority vectors is 15.85%, calculated with the common formula: e=Abs(w*-w)/w, for each coordinate and then adding over the coordinates.

But, Saaty's consistency index says that CR=95% (or 5% of inconsistency), which is a tolerable limit for a 3x3 PC matrix. Hence, the critic claims that Saaty's consistency index is wrong.

The Response:

The critic has two important misunderstandings:

First: The CR (the Saaty's index of consistency) come from the eigenvalue-eigenvector problem, so it is a systemic approach (do not care in any particular comparison). [4,5]

Second: The possible error should be measured by its final result (the resulting metric), not in the prior or middle steps.

The Explanation:

The first misunderstanding is explained it by itself (systemic approach).

For the second one, before any calculation, we need to understand what kind of numbers are we dealing with (in what environment we are working), because for errors and deviations is not the same to be close to a big priority than to a little one. This is a weighted environment and the measure of the closeness (proximity) and possible errors has to consider this situation.

We must work in the order topology domain to correct measure the closeness on this environment. To do this task correctly two aspects of the information have to be considered, the intensity (the weight or priority) and the degree of deviation between the two priority vectors (the projection between the vectors). The index that take good care of these two factors simultaneously is the compatibility index *G*.

Summarizing, the vectors of correct and perturbed metric are:

Correct metric (priority vector) :	0.3333	0.3333	0.3333
Perturbed or approximated metric (priority vector) :	0.4126	0.3275	0.2599

The basic question here is how close is the approximated metric to the correct metric?

Evaluating G(Correct-Perturbed), the G value obtained is: 85.72%, which in numerical terms represent almost compatible metrics. (See Table 2: Ranges of compatibilities and its meaning).

As explained at the end of point 5, G=90% is a threshold to consider two priority vectors as compatible vectors. Also, G=85% is an acceptable lower limit value. Hence**,** the two metrics are relatively close (close enough considering that they are not physical measures).

It is important to say that the same exercise was performed from 4x4 to 9x9 matrices that is, putting a value (n-1) in the position cell (1, n), (n= matrix dimension), obtaining even better results for the compatibility index G. As it is show in TABLE 5 below.

Inconsistency												
5%	3x3	0,333333333	0,333333333	0,333333333	1							
	(1,3)==>2	0,4126	0,3275	0,2599	1							
		0,30131416	0,324634375	0,231272015		**85,7%**						
6%	4x4	0,25	0,25	0,25	0,25	1						
	(1,4)==>3	0,331	0,2407	0,2407	0,1888	1,0012						
		0,219410876	0,23622298	0,23622298	0,16569088		**85,8%**					
6%	5x5	0,2	0,2	0,2	0,2	0,2	1					
	(1,5)==>4	0,277	0,1906	0,1906	0,1906	0,151	0,9998					
		0,172202166	0,1861209	0,1861209	0,1861209	0,1325025		**86,3%**				
5%	6x6	0,1667	0,1667	0,1667	0,1667	0,1667	0,1667	1				
	(1,6)==>5	0,2392	0,1582	0,1582	0,1582	0,1582	0,1279	0,9999				
		0,14139725	0,15418172	0,15418172	0,15418172	0,15418172	0,11302523		**87,1%**			
4%	9x9	0,111111111	0,111111111	0,111111111	0,111111111	0,111111111	0,111111111	0,11111111	0,11111111	0,11111111	1	
	(1,9)==>7	0,1717	0,1055	0,1055	0,1055	0,1055	0,1055	0,1055	0,1055	0,0897	0,9999	
		0,091506863	0,102836125	0,102836125	0,102836125	0,102836125	0,102836125	0,10283613	0,10283613	0,08105741	0,89241714	**89,2%**

TABLE 5: Compatibility indices for perturbed matrices from range 3x3 to 9x9

This outcome was not a surprise, since it comes from a matrix built within an intrinsic systemic behavior. The pair comparison process in the matrix produces highly related elements among them. When searching for the equilibrium point of the matrix (the eigenvector or the weighted metric of the matrix), this process of relations and interconnections can be perceived as a growing complex system as graph theory clearly shows. Thus, the analysis of the quality of the consistency index must be done considering this relevant fact (complex system), with many connections and redundancies. By the way, the redundancies are necessary because it gives more precision and credibility to the system (any system without redundancy is a vulnerable system). In addition, these (necessary) redundancies give more stability to the system, because it allows having a cell in the matrix with a very bad pair comparison value. For instance, in Table 5, for the case of 9x9 matrix in position (1, 9), there is a 8 instead of 1, that is a "very large error" of 800%, and spite of that, we still got a healthy outcome of 89.2% for compatibility index G (even better than the rest of G values). Even more, when performing the hypothetic case for n=15 (15x15 matrix), with a value of 14 in the position cell (1, 15), the outcome for compatibility is 99.96% (almost 100% of compatibility). Thus, the compatibility trend clearly shows that the divergence in the value of cell (1, n), (or any other cell by the way), is not producing any decay in the quality of the generated metric. By the way, the consistency index for this case was 3%, a very good index.

Finally, when a system allow redundancies it has the capacity to receive new information that may or may not be consistent with the old one. This characteristic allow the system to evolve, connecting old and new data in a peaceful way.

By the other hand, if we leave "*n*" (the size of the matrix) free to grow beyond 9, then the acceptable error in Saaty's consistency index diverge (goes to infinity). It means that you can put in the PC matrix a number as bigger as you like and still get an acceptable ratio of consistency. This abnormal behavior for RC is also revealed by the compatibility index G. As "n" and the pair comparison value increase, G decrease making the priority vector (the final metric) more and more incompatible with the reference vector. For instance, in a 15x15 PC matrix the value for cell (1, 15) can be as large as 50 (5000% of "error"), a very large value (beyond one order of magnitude) and still be consistent (10%). In this case, the output value for G is 79.7%, which according to Table 2 is not an acceptable value for a quality test.

However, what means to leave "*n*" free to grow in a weighted environment? It means that our reference vector or metric defined by: {1/n, 1/n,…,1/n} become (or tend) to {0, 0,..,0}, the null vector. The null vector (or zero vector) is not a point of reference for anything in weighted environment *(using this vector as reference point is like dividing by zero in a mathematical demonstration*). Thus, the useful mathematical concept of limit analysis behavior is not applicable here.

There are many important conclusions from this example:

1. Is not true that Saaty's consistency index is irremediable and mathematically unsound. The proof based on "n" ➔ ∞ and still getting a consistent PC matrix, which, by the way, means make the error in the element (1, n) as big as you want and still have consistency, is not a good argument. Because, the nxn matrix with n➔∞, in decision-making space is represented by the metric: (1/n, 1/n…, 1/n) = (0, 0..., 0), which is not a useful metric to make any valid mathematical proof, the null metric is not defined in the absolute ratio scale. Besides, the null vector does not define any metric space, (neither in the decision making space).
2. When possible, all pair comparisons in the matrix have to be done, and all of the pair comparisons have to be taking it into account, the rights and the wrongs, (which by the way are indistinguishable), to correctly assess consistency and priority (the weighted metric of the matrix). Here is where the eigenvector operator (and its principal eigenvalue) perform it the best.
3. Is not valid the analysis of the behavior of an isolated element to characterize all the system (this would be a kind of basic mechanical analysis). The PC matrix represents a highly related system. Thus, you cannot evaluate a complex system behavior (the PC matrix) only by the behavior of one of his elements; there are redundancies that are not well captured in the one isolated element analysis.

From the last paragraph of this example, we may conclude:

4. If you want to have a representative consistency index (it means without a very large bad comparison), you should never go beyond n=9 (9x9 matrices) in any PC matrix. By the way, this is aligned with Axiom 2 of AHP, which is keeping the homogeneity not beyond one order of magnitude between the elements to be compared.
5. Finally, it seems that the threshold of 10% of Saaty's consistency index could be too lax for the general case. This threshold should not go beyond 5% value for 3x3 matrices (as it is now), and in a value not beyond 6% in matrices of superior order instead of the current 10%, if we want to keep an acceptable level of compatibility, this also depend of the kind of problem to be solve. However, we think that more investigation and numerical tests should be carry on in this line of research.

Of course, better consistency (100% for instance), can always be achieved. The question is: do we really obtain a better result when being totally consistent?

The answer is: probably no. Because, in real problems we never have the "real" answer (the true metric to use as reference). Experience shows that pursuing consistent metrics per se, may provide less sustained results.

For instance, in the presented problem one could answer that: A-B=2, A-C=2, and B-C=1, as showed in FIGURE 5, and he/she would be totally consistent, but consistently wrong.

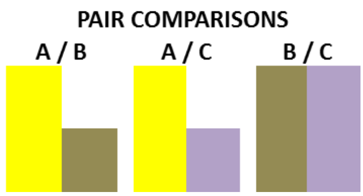


FIGURE 5: total consistent pairwise matrix

By the way, in FIGURE 5 the new priority vector is w**= (0.5, 0.25, 0.25), with CR=0 (totally consistent), and G= 71.5%, which means not compatible vectors (low compatibility). Thus, a totally consistent metric is incompatible with the correct result.

So, at the end it is better to be approximately correct than consistently wrong.

(The consistency index is just a thermometer not a goal).

It is also important to bring to the conversation the fact that the quality of the metric of any PC matrix is not found in some specific PC judgment of the matrix. It should be found in its final interaction, which means after that all the relations and redundancies have played its part in its search of the equilibrium point of the system (the main eigenvector). So, define the quality of the metric that come from a PC-matrix directly from the matrix values (as presented in this criticism example) is not a good idea.

6. CONCLUSIONS

The study bring to MCDM academy an index to measure closeness in weighted environment, a very necessary index to be able to define if two different measurement space (two different system value for instance), are compatible or not, are able to be used as a common decision metric. Also if some pattern of behaviour is compatible with another one already knew, which can be contrasted. If so, we have a powerful tool to recognize elements in a social scheme under a scientific frame.

The formula introduced in the paper (the G index), has been tested against different compatibility indices and distance norms, showing a better performance in many different tested cases. Later, G was compared with other similar indices coming from other fields of science, like statistics and biology. The index from biology (the Jaccard index) has a particular interest, since it comes from the set theory (very close to topology analysis). In fact, in some way G index can be seen as a generalization of Jaccard index in the order topology domain.

Later, was built a threshold for the G index, an indispensable element to complement G. The threshold was tested in many different cases, using different systems of approach: statistical approach, sensibility approach, pattern recognition approach, music analysis or frequency recognition, and testing control (using results already knew to compare with).

Then, G was applied in two different examples, the first about the peculiar notion that; equal ordinal rank means compatibility (two persons with the same rank order are necessarily compatible), that sentence can be a totally wrong conclusion. Even the inverse sentence (two compatible people have the same rank order) in a weighted environment is not necessarily true. Compatibility is a cardinal issue not ordinal, and this is a relevant fact. Even under the light of the old discussion about rank reversal, where to preserve the ordinal rank of a set of alternatives became something almost religious. G shows that two metrics can be close no matter their exactly rank order (indeed, two metrics with different rank order can be more close than other two with the same order rank). At first glance, this conclusion it may seems a little contra intuitive. However, it is based in the more familiar idea that if we are interested to be similar to some specific profile in a global way, then it is better to be close (be similar) to few relevant criteria of the profile, than to be close to many low relevant criteria of the same profile. (Compatibility is not a fact of how many, but of how much).

The following example in the paper shows that systemic behaviour cannot be analysed by its elements in a separate way. The PC matrix is a system and the pair comparison judgments are its elements. G index shows that you may have one poor comparison (very poor indeed), and still have an acceptable quality metric as a result (an acceptable priority vector). The relevant conclusions from this example were presented at the end of the example itself, (in case the reader wishes to reread it again), and it is also a response or rebuttal to the failed criticism about that Saaty's consistency index is useless and mathematically unsound.

There is huge number of other possible applications of G index in many different fields. Some examples of this are: On Medicine, measuring the degree of matching (proximity) between patient and disease diagnose profiles. On Buyers-Seller matching profiles; measuring the degree of matching between house buyers and sales project. On Group Decision Making; measuring how close are two (or more) different value systems. On Quality Tests, measuring what MCDM decision method can builds a better metric. On Agricultural; measuring the proximity between the cultivate plants against a healthy plant (based on its micro & macronutrients) and selecting the best nutrient seller. On Shiftwork Prioritization; measuring how close are the different views among the different stakeholders (Workers view, Company view, Community view). On Company Social Responsibility (CSR); measuring how close are the different views among the different stakeholders (Economic, Environmental and Social view).

Note: All the examples mentioned above come from real cases of application.

7. ACKNOWLEDGMENTS

In memory of Dr. Thomas L. Saaty. Professor, Master and Friend. We will remember you forever.

8. KEY REFERENCES

GARUTI, C. (2012): "MEASURING IN WEIGHTED ENVIRONMENTS: MOVING FROM METRIC TO ORDER TOPOLOGY". Santiago, Chile: Universidad Federico Santa Maria (Book, 72pp).

GARUTI, C. (2014a): "MEASURING IN WEIGHTED ENVIRONMENTS". International Journal of AHP (IJAHP) 2014. (Article).

GARUTI, C. (2014b): "COMPATIBILITY OF AHP/ANP VECTORS WITH KNOWN RESULTS". Presentation of a suggested new index of compatibility in weighted environments. ISAHP2014 International Symposium of the Analytic Hierarchy Process.

JACCARD P. (1901): "DISTRIBUTION DE LA FLORE ALPINE DANS LE BASSIN DES DRANSES ET DANS QUELQUES RÉGIONS VOISINES". Bulletin de la Société Vaudoise des Sciences Naturelles 37, 241-272

MAHALANOBIS, P.C. (1936): "ON THE GENERALIZED DISTANCE IN STATISTICS", Proceedings of the National Institute of Science of India 12 49-55.

SAATY, L.T. (2001): "THE ANALYTIC NETWORK PROCESS: DECISION MAKING WITH DEPENDENCE AND FEEDBACK". (Book, 376pp).

SAATY, L. T. (2010): "GROUP DECISION MAKING: DRAWING OUT AND RECONCILING DIFFERENCES". Pittsburgh: RWS. (Book, 385pp).